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On Explicitly Solvable Gradient Systems of Moser–Karmarkar Type

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A generalization and a classification of explicitly solvable gradient systems which appear in linear programming and neurodynamics are studied, It is shown that nonlinear dynamical systems of Moser–Karmarkar type generically take both a Lax representation and a gradient form. A related optimization problem on a simplex is discussed. An integration procedure and a complete description of the invariant manifolds and the phase portrait of the dynamical systems of Moser–Karmarkar type are also investigated. © 1997 Academic Press

1. INTRODUCTION

In his fundamental paper [3], R. W. Brockett introduced the double-bracket equation and showed that it is related to many problems of applied mathematics. It is now generally believed that completely integrable Hamiltonian systems that show gradient-like behavior on common level surfaces of their integrals admit a Lax representation in the form of a double-bracket equation (cf. [4]). It is certainly true for Moser's dynamical

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system [1] and its Lie-algebraic generalizations [2], for completely integrable gradient flows arising in geometric treatment of information spaces [19, 22], and for dynamical systems that solve linear programming problems [7, 8]. It is also true for Karmarkar's dynamical system [20, 5] arising in connection with polynomial-time algorithms for solving linear programming problems [15, 16]. It is usually possible to completely describe the phase portraits of such dynamical systems. Explicitly solvable gradient flows also play a significant role in neurodynamics and learning [21], in mathematical biology [12], and in the deterministic approximation of a random collision model [13]. Very recently completely integrable gradient flows of [19, 22] found applications to the maximum likelihood estimation and Boltzmann machines [9].

In the present paper we embed both Moser's and Karmarkar's dynamical systems into a family of explicitly solvable gradient flows that are parametrized by a real parameter m. Explicit formulas for solutions of these dynamical systems as well as a complete description of their phase portraits are presented. There are two exceptional values of the parameter m: $m = 1$ (Moser's dynamical system) and $m = 3$. We will show that the changes of the topological structure of the phase portrait occur exactly at these two values of the parameter m. The Lax representation in the double-bracket form is constructed for all values of m except for $m = 3$.

Consider the nonlinear dynamical systems

$$\frac{dx_j}{dt} = -c_j x_j^m + x_j \sum_{k=1}^{n} c_k x_k^2, \qquad j = 1, 2, \dots, n, \tag{1}$$

where c_j are nonzero real constants and m is an arbitrary real parameter. Throughout this paper we are interested in the solutions lying in the positive orthant, namely, $x = (x_j) > 0$. When $m = 1$ and c_j are pairwise distinct, (1) is just Moser's dynamical system [17]. If c_j are arbitrary, (1) with $m = 1$ appears in neurodynamics [21]. Moreover, setting $y_j = x_j^2$ in (1) with $m = 1$ we can derive a Lotka–Volterra equation which describes a competing n species. When $m = 2$, (1) coincides with Karmarkar's dynamical system [16]. We call (1) the dynamical systems of Moser–Karmarkar type.

In the next section we present a Lax representation of (1) when $m \neq 3$. A potential function is also found which is useful for expressing (1) in a gradient form. A related optimization problem on a simplex is discussed. In Section 3, an integration procedure for (1) is considered. It is shown that (1) is reduced to a single differential equation when $m \neq 3$. In Section 4, a complete description of the phase portrait of (1) is established in terms of invariant manifolds. Moser's system where $m = 1$ is exceptional here.

When $m < 3$ and $m \neq 1$, there exist many stable fixed points. The behavior of a solution is investigated in two cases, where $1 < m < 3$ and where $m < 1$, respectively. In Section 5, a slight modification of (1) with cubic nonlinearity, i.e., $m = 3$, is shown to be an explicitly solvable gradient system having a Lax representation.

2. LAX REPRESENTATION AND GRADIENT FORM

We first look for invariant manifolds and first integrals of (1) in the form $\sum_{j=1}^{n} x_j^l$ for some constant l. Differentiating this and using (1) we have

$$\frac{d}{dt} \sum_{j=1}^{n} x_j^l = -l \sum_{j=1}^{n} c_j x_j^{m+l-1} + l \sum_{j=1}^{n} x_j^l \sum_{k=1}^{n} c_k x_k^2.$$

This implies that if the initial value satisfies $\sum_{j=1}^{n} x_j^{3-m}(0) = 1$, then for any $t \in \mathbf{R}$, $\sum_{j=1}^{n} x_j^{3-m}(t) = 1$ except for the case where $m = 3$. When $m = 1$, (1) is reduced to a Lotka–Volterra equation by setting $y_j = x_j^2$. The quantity $\sum_{j=1}^{n} y_j$ is known to be one of the first integrals of a cyclic Lotka–Volterra equation (cf. [14]). Here we introduce the invariant manifold

$$P_m = \left\{x \geq 0 \mid x_1^{3-m} + x_2^{3-m} + \cdots + x_n^{3-m} = 1, m\right\}, \qquad m \neq 3. \tag{2}$$

On the other hand consider the case where $m = 3$. If $n = 2$ and $c_1 + c_2 = 0$, then (1) admits the first integral $x_1^2 + x_2^2 = c^2$ for some positive constant c. Let us set

$$P_3 = \left\{x \geq 0 \mid x_1^2 + x_2^2 = c^2, c > 0\right\}. \tag{3}$$

We have proved

LEMMA 2.1. *The nonlinear dynamical systems* (1) *have the invariant manifold* (2) *for* $m \neq 3$ *and the first integral* (3) *for* $m = 3$ *with* $n = 2$ *and* $c_1 + c_2 = 0$.

Next we discuss a Lax representation of the dynamical systems of Moser–Karmarkar type (1) on its phase space

$$\operatorname{int}(P_m) = \left\{x \in P_m \mid x_i > 0, i \in [1, n]\right\}, \qquad m \neq 3, \tag{4}$$

where $[1, n] = \{1, 2, \ldots, n\}$. Define

$$\begin{aligned} L &= \begin{pmatrix} x_1^{3-m} & (x_1x_2)^{(3-m)/2} & \ldots & (x_1x_n)^{(3-m)/2} \\ (x_2x_1)^{(3-m)/2} & x_1^{3-m} & \ldots & (x_2x_n)^{(3-m)/2} \\ \vdots & \vdots & \ddots & \vdots \\ (x_nx_1)^{(3-m)/2} & (x_nx_2)^{(3-m)/2} & \ldots & x_n^{3-m} \end{pmatrix}, \\ D &= \frac{3-m}{2} \begin{pmatrix} c_1x_1^{m-1} & & & \mathbf{0} \\ & c_2x_2^{m-1} & & \\ & & \ddots & \\ \mathbf{0} & & & c_nx_n^{m-1} \end{pmatrix}. \end{aligned} \tag{5}$$

After a tedious calculation we derive from (1) and (5)

$$\frac{dL}{dt} = \frac{3-m}{2}\left((x_ix_j)^{(3-m)/2}\left(2\sum_{l=1}^{n} c_lx_l^2 - c_ix_i^{m-1} - c_jx_j^{m-1}\right)\right),$$

$$\begin{aligned} [[L, D], L] = \frac{3-m}{2}\Bigg((x_ix_j)^{(3-m)/2} \\ \times\left(2\sum_{l=1}^{n} c_lx_l^2 - \left(c_ix_i^{m-1} + c_jx_j^{m-1}\right)\sum_{k=1}^{n} x_k^{3-m}\right)\Bigg), \end{aligned}$$

where $[A, B]$ denotes the commutator of A and B, $[A, B] = AB - BA$. Let us assume $\sum_{k=1}^{n} x_k^{3-m}(0) = 1$. Then using Lemma 2.1 we prove that (1) is expressed in the double-bracket Lax form

$$\frac{dL}{dt} = [[L, D], L]. \tag{6}$$

When $m = 1$ and $m = 2$, (6) leads to the Lax representations of Moser's system [1, 21] and Karmarkar's system [20, 5], respectively. When $m = 2$ and $c_i = 2$, (6) is a special case of an equation of Lax type which solves a matrix eigenvalue problem [18]. A big advantage of Lax representations is that the trace of power of L is automatically a first integral. In this case trace(L) coincides with $\sum_{i=1}^{n} x_i^{3-m}$; see (2). Throughout this paper we assume

$$\sum_{k=1}^{n} x_k^{3-m}(0) = 1$$

when $m \neq 3$.

For the case where $m = 3$ with $n = 2$ and $c_1 + c_2 = 0$ we set

$$L = \begin{pmatrix} x_1^2 & x_1 x_2 \\ x_2 x_1 & x_2^2 \end{pmatrix}, \qquad D = \frac{\lambda}{c^2} \begin{pmatrix} x_1^2 & 0 \\ 0 & -x_2^2 \end{pmatrix}, \tag{7}$$

where $\lambda = c_1 = -c_2$. Using Lemma 2.1 we can check that (1) is equivalent to

$$\frac{dL}{dt} = [[L, D], L]. \tag{8}$$

It is concluded that

THEOREM 2.1. *The dynamical systems of Moser–Karmarkar type* (1) *admit the Lax representation* (6) *when* $m \neq 3$ *and* (8) *when* $m = 3$ *with* $n = 2$ *and* $c_1 + c_2 = 0$.

We now discuss the problem how to express (1) in the gradient form $dx/dt = -\mathrm{grad}_g U$ for a potential function $U = U(x)$ relative to a metric $g = (g_{ij}(x))$. If we find U and g, then it follows that $dU/dt = -\|\mathrm{grad}_g U\|^2 \leq 0$ on the trajectory of (1), where $\| * \|$ is the norm defined by the metric g. Consequently, the dynamical systems (1) may be related to a certain optimization problem as Moser's system and Karmarkar's system are.

Our result is as follows. Let us define

$$\begin{gathered} U = \frac{1}{2} \sum_{k=1}^{n} c_k x_k^2 \left(\sum_{i=1}^{n} x_i^{3-m} \right)^{2/(m-3)}, \\ (g_{ij}) = \begin{pmatrix} x_1^{1-m} & & 0 \\ & \ddots & \\ 0 & & x_n^{1-m} \end{pmatrix} \end{gathered} \tag{9}$$

for the case $m \neq 3$. By projecting $\partial U/\partial x_j$ to $\sum_{j=1}^{n} x_j^{3-m} = 1$ we have

$$\left. \frac{\partial U}{\partial x_j} \right|_{\sum_{j=1}^{n} x_j^{3-m} = 1} = c_j x_j - x_j^{2-m} \sum_{k=1}^{n} c_k x_k^2.$$

Thus it is not hard to see that (1) for $m \neq 3$ is expressed as the gradient form

$$\frac{dx_i}{dt} = -\sum_{j=1}^{n} g^{ij} \left. \frac{\partial U}{\partial x_j} \right|_{\sum_{k=1}^{n} x_k^{3-m} = 1}, \tag{10}$$

where g^{ij} are elements of the inverse of the positive definite matrix $g = (g_{ij})$. When $m = 1$ the potential U of (9) originally appears in Moser [17]. When $m = 2$ the metric $(g_{ij}) = (x_i^{-1}\delta_{ij})$ is known as the Shahshahani metric in mathematical biology [12]. It is also used in [6] for solving optimization problems on an arbitrary polytope. Here the dynamical system (1) for $m = 2$ is a special case of Fisher's fundamental equation in natural selection [10]. The Shahshahani gradient form (10) is corresponding to M. Kimura's maximal principle [12], which states that a change occurs in such a way that the increase in U is maximal. The metric g in (9) for $m = 2$ was found in [20, 5] for Karmarkar's system independently. On the other hand, the metric $(g_{ij}) = (x_i^{-2}\delta_{ij})$ was introduced by Karmarkar [15, 16]. The Lax form (6) and the generalized Shahshahani gradient (10) are directly related to each other as follows. Define an inner product by using the metric g. Note that (6) describes an adjoint orbit of an orthogonal group through $L(0)$. It is known (cf. [4]) that the gradient system of U with respect to the inner product on the adjoint orbit always takes double-bracket Lax form.

For the exceptional case where $m = 3$ we can express (1) as a gradient form provided that $n = 2$ and $c_1 + c_2 = 0$. Let us set

$$U = \frac{1}{4}\,\frac{\sum_{k=1}^{2} c_k x_k^2}{\sum_{i=1}^{2} x_i^2}, \qquad (g_{ij}) = \frac{1}{c^2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{11}$$

The gradient systems generated by U and (g_{ij}) are therefore

$$\frac{dx_i}{dt} = -\sum_{j=1}^{2} g^{ij} \left.\frac{\partial U}{\partial x_j}\right|_{\sum_{k=1}^{2} x_k^2 = c^2}. \tag{12}$$

We have shown

THEOREM 2.2. *The dynamical systems* (1) *are expressed in the gradient form* (10) *when* $m \neq 3$ *and* (12) *when* $m = 3$ *with* $n = 2$ *and* $c_1 + c_2 = 0$.

Finally, in this section we show how the dynamical systems of Moser–Karmarkar type (1) are related to an optimization problem on a simplex. For this purpose we introduce new variables $z_i(t)$ by

$$z_i(t) = x_i^{3-m}(t), \qquad m \neq 3.$$

The dynamical systems (1) on $\operatorname{int}(P_m)$ are transformed into that on $\operatorname{int}(P_m^{(z)}) = \{z_i \mid \sum_{i=1}^{n} z_i = 1, z_i > 0\}$, an interior of the simplex, which are gradient systems of $U = \frac{1}{2}\sum_{k=1}^{n} c_k z_k^{2/(3-m)}(\sum_{j=1}^{n} z_j)^{2/(m-3)}$ restricted to $\operatorname{int}(P_m^{(z)})$ relative to the Shahshahani metric $(g_{ij}) = (z_i^{-1}\delta_{ij})$. Hence (1) solves the corresponding optimization problem for $U = \frac{1}{2}\sum_{k=1}^{n} c_k z_k^{2/(3-m)}$

on the simplex. In particular, Moser's dynamical system where $m = 1$ is related to a linear programming problem for minimizing $U = \frac{1}{2}\sum_{k=1}^{n} c_k z_k$ subject to $\sum_{i=1}^{n} z_i = 1$. In [1] Bloch pointed out that Moser's dynamical system also solves the same linear programming problem as a gradient system with respect to a Riemannian metric on the manifold of orthogonal matrices. For the case of a general polytope see [6]. The picture discussed here sheds new light on the connection between Moser's system and linear programming.

Alternatively consider the change of variables

$$y_i(t) = x_i^{(3-m)/2}(t), \qquad m \neq 3.$$

In new variables (1) takes the form

$$\frac{dy_i}{dt} = \frac{3-m}{2}\left(-c_i y_i^{(m+1)/(3-m)} + y_i\left(\sum_{k=1}^{n} c_k y_k y_k^{(m+1)/(3-m)}\right)\right) \tag{13}$$

and the invariant manifold (2) will take the form

$$\sum_{k=1}^{n} y_k^2 = 1. \tag{14}$$

Observe, first of all, that (13) is the gradient relative to the Euclidean metric restricted to the sphere (14) of the function

$$-\frac{(3-m)^2}{8}\sum_{k=1}^{n} c_k y_k^{4/(3-m)}. \tag{15}$$

The following simple lemma is a particular case of [6] and can be proved by a direct computation.

PROPOSITION 2.1. *Let*

$$\begin{gathered}\frac{dy}{dt} = r(y) - \langle y, r(y)\rangle y,\\ r(y) = \left(v_1(y)y_1, \ldots, v_n(y)y_n\right)^{\top}, \qquad y \in \mathbf{S}^{n-1}\end{gathered} \tag{16}$$

for the smooth functions

$$v_i(y) = -\frac{3-m}{2} c_i y_i^{2(m-1)/(m-3)}.$$

Then for $L(y) = yy^{\top}$ *we have*

$$\frac{dL(y)}{dt} = [[D, L(y)], L(y)], \qquad D = \operatorname{diag}(v_i(y), \ldots, v_n(y)) \tag{17}$$

along solutions to (16).

We can rewrite the Lax representation (17) to (6) in terms of the original variables x_i.

3. INTEGRATION PROCEDURE

In this section we discuss an integration procedure of the dynamical systems of Moser–Karmarkar type (1) with the help of the invariant manifold P_m given in Lemma 2.1. The exceptional cases here are $m = 1$ and $m = 3$.

Let us assume $m \neq 1, 3$. Define

$$\begin{aligned} S_j &= x_j^{1-m}, \qquad q(x) = \sum_{k=1}^{n} c_k x_k^2, \\ T(t) &= \exp\left((m-1) \int_0^t q(x(s))\, ds \right). \end{aligned} \tag{18}$$

For the dynamical systems (1) we derive

$$\frac{1}{1-m} \frac{dS_j}{dt} = -c_j + S_j q(x).$$

Since $dT/dt = (m-1)q(x)T$, we obtain

$$\frac{d(S_j T)}{dt} = (m-1) c_j T. \tag{19}$$

Noting that $S_j(0)T(0) = x_j^{1-m}(0)$, we integrate (19) to $S_j T = x_j^{1-m}(0) + (m-1)c_j P$, where $P(t) = \int_0^t T(s)\, ds$. It follows from the definition (18) that

$$x_j(t) = \left(\frac{dP/dt}{x_j^{1-m}(0) + (m-1)c_j P} \right)^{1/(m-1)}. \tag{20}$$

The indeterminant $P(t)$ in (20) is given by solving the single equation

$$\frac{dP}{dt} \left(\sum_{j=1}^{n} \left(x_j^{1-m}(0) + (m-1)c_j P \right)^{(m-3)/(m-1)} \right)^{(m-1)/(m-3)}, \tag{21}$$

which comes from the invariant manifold (2) and (20). It is concluded that

THEOREM 3.1. *The solution to* (1) *is given by integrating* (21) *through* (20) *when* $m \neq 1, 3$.

In some cases (21) takes a relatively simple form. For example, let $m = \frac{5}{3}$. Then (21) reads

$$\frac{dP}{dt} = \left(\sum_{j=1}^{n} \frac{1}{\left(x_j^{-2/3}(0) + \frac{2}{3} c_j P \right)^2} \right)^{-1/2}.$$

Alternatively, when $m = 2$, the Karmarkar case, (21) is

$$\frac{dP}{dt} = \left(\sum_{j=1}^{n} \frac{1}{x_j^{-1}(0) + c_j P} \right)^{-1}.$$

Thus we obtain

$$\sum_{j=1}^{n} \frac{1}{c_j} \frac{d \log\left(x_j^{-1}(0) + c_j P \right)}{dt} = 1.$$

Observe that this equation can be easily integrated. Thus $P(t)$ can be obtained by solving an algebraic equation without integrating the differential equation (21). See Theorem 3 in [20].

Next let us consider the case where $m = 1$. We present an integration procedure which is slightly differential from that in [17]. Since

$$\frac{d \log x_j}{dt} = -c_j + q(x),$$

where $q(x) = \sum_{k=1}^{n} c_k x_k^2$, obtain

$$x_j(t) = x_j(0) \exp(-c_j t) \exp\left(\int_0^t q(x(s))\, ds \right). \tag{22}$$

Inserting (22) into $\sum_{k=1}^{n} x_k^2(t) = 1$ we can derive the expression of solution

$$x_j(t) = \frac{x_j(0) \exp(-c_j t)}{\left(\sum_{k=1}^{n} x_k^2(0) \exp(-2 c_k t) \right)^{1/2}}. \tag{23}$$

Finally, we discuss the case where $m = 3$. Let us assume $n = 2$ and $c_1 + c_2 = 0$. There is the first integral $x_1^2(t) + x_2^2(t) = c^2$ from Lemma 2.1. Set $x_1 = c \cos \theta$ and $x_2 = c \sin \theta$. We then see that θ satisfies $2\, d\theta/dt = \lambda c^2 \sin(2\theta)$, where $\lambda = c_1$. Integrating this equation we have

$$\begin{aligned} x_1(t) &= \frac{c}{\left(\exp(2 \lambda c^2 t - \mu) + 1 \right)^{1/2}}, \\ x_2(t) &= \frac{c}{\left(\exp(-2 \lambda c^2 t + \mu) + 1 \right)^{1/2}}, \end{aligned} \tag{24}$$

where $\mu = 2 \log(x_1(0)/x_2(0))$. It follows from (24) that $(x_1(\infty), x_2(\infty)) = (0, c)$ if $c_1 > c_2$ and $(x_1(\infty), x_2(\infty)) = (c, 0)$ if $c_1 < c_2$ as $t \to \infty$. When $c_1 = c_1$, any initial value becomes a fixed point of trajectory, namely, $(x_1(0), x_2(0)) = (x_1(\infty), x_2(\infty))$.

4. THE PHASE PORTRAIT

In this section using the method of [5] we describe the phase portrait of the dynamical systems of Moser–Karmarkar type (1) for the case $m < 3$ on the corresponding invariant manifold P_m defined in (2). Assume again that $c_i \neq 0$ for all i. Denote by $J_+ = \{i \in [1,n] \mid c_i > 0\}$ and by $J_- = \{i \in [1,n] \mid c_i < 0\}$, where $[1,n] = \{1,2,\ldots,n\}$. Let Γ be the set of all nonempty subsets J in $[1,n]$ such that $J \subset J_-$ or $J \subset J_+$.

PROPOSITION 4.1. *For any $m < 3$, $m \neq 1$, there is a one-to-one correspondence between nonempty subsets $J \subset \Gamma$ and fixed points $x(J)$ of (1) in P_m. Namely,*

$$x_i(J) = \left(\frac{|\alpha(J)|}{|c_i|} \right)^{1/(m-1)}, \qquad i \in J, \tag{25}$$

$$x_i(J) = 0, \qquad i \notin J, \tag{26}$$

where

$$\alpha(J) = \epsilon(J) \left(\sum_{i \in J} |c_i|^{(m-3)/(m-1)} \right)^{(m-1)/(m-3)} \tag{27}$$

with $\epsilon(J) = 1$ for $J \subset J_+$ and $\epsilon(J) = -1$ for $J \subset J_-$.

Proof. An easy computation shows that for any $J \subset \Gamma$ the point $x(J)$ described by (25)–(27) belongs to P_m and is a fixed point of (1). Denote by $J \subset [1,n]$ the set of indices i such that $x_i > 0$. We have from (1)

$$c_i x_i^{m-1} = q(x), \qquad i \in J, \tag{28}$$

where $q(x)$ is defined in (18). Set $\alpha(J) = q(x)$. Since $c_i \neq 0$ for all i, $\alpha(J) \neq 0$. If $\alpha(J) > 0$, then by (28), $c_i > 0$ for all $i \in J$ and $J \subset J_+$. If $\alpha(J) < 0$, then $c_i < 0$ for all $i \in J$ and $J \subset J_-$. Using (2) and (28) we arrive at (25), (27). ∎

We give here an example where $m = 2$, $n = 3$, $c_i > 0$ for all i. There are seven fixed points on P_m. They are

$$x_a(J) = (1,0,0), \qquad x_b(J) = (0,1,0), \qquad x_c(J) = (0,0,1),$$

$$x_s(J) = \left(\frac{c_2}{c_1 + c_2}, \frac{c_1}{c_1 + c_2}, 0 \right), \qquad x_e(J) = \left(\frac{c_3}{c_1 + c_3}, 0, \frac{c_1}{c_1 + c_3} \right),$$

$$x_f(J) = \left(0, \frac{c_3}{c_2 + c_3}, \frac{c_2}{c_2 + c_3} \right),$$

$$x_g(J) = \frac{1}{c_1 c_2 + c_2 c_3 + c_3 c_1} (c_2 c_3, c_3 c_1, c_1 c_2).$$

Recall that we denote by $\operatorname{int}(P_m)$ the set

$$\{x \in P_m : x_i > 0, i \in [1, n]\}, \qquad m \neq 3.$$

PROPOSITION 4.2. *Let $x(t)$ be a solution to* (1) *in* $\operatorname{int}(P_m)$. *Then*

$$\frac{dq(x(t))}{dt} = -2 \sum_{i=1}^{n} x_i^{3-m}(t)\big(q(x(t)) - c_i x_i^{m-1}(t)\big)^2. \tag{29}$$

In particular, $q(x(t))$ is a monotonically decreasing function along the corresponding solution $x(t)$ to (1).

Proof. Indeed, using (1)

$$\frac{dq(x(t))}{dt} = 2 \sum_{i=1}^{n} c_i x_i(-c_i x_i^m + x_i q) = 2\left(q^2 - \sum_{i=1}^{n} c_i^2 x_i^{m+1}\right).$$

On the other hand,

$$\begin{aligned} -2 &\sum_{i=1}^{n} x_i^{3-m}\big(q - c_i x_i^{m-1}\big)^2 \\ &= -2\left(q^2 - 2 \sum_{i=1}^{n} \big(x_i^{3-m} q c_i x_i^{m-1}\big) + \sum_{i=1}^{n} c_i^2 x_i^{2m-2+3-m}\right) \\ &= 2\left(q^2 - 2 \sum_{i=1}^{n} c_i^2 x_i^{m+1}\right). \end{aligned}$$

Here we used (2). ∎

PROPOSITION 4.3. *Suppose that $x(t)$ is a solution to* (1) *in* $\operatorname{int}(P_m)$. *Then for any indices $i, j \in [1, n]$ we have*

$$\begin{aligned} &\frac{d}{dt}\big(c_i x_i^{m-1}(t) - c_j x_j^{m-1}(t)\big) \\ &\qquad = (m-1)\big(c_j x_j^{m-1}(t) - c_i x_i^{m-1}(t)\big) \\ &\qquad\qquad \times \big(c_j x_j^{m-1}(t) + c_i x_i^{m-1}(t) - q(x(t))\big). \end{aligned} \tag{30}$$

Proof. A direct computation. ∎

Let $V_{ij}^0(m)$, $V_{ij}^{\pm}(m)$ be defined as

$$\begin{aligned} V_{ij}^0(m) &= \big\{x \in \operatorname{int}(P_m) \mid c_i x_i^{m-1} = c_j x_j^{m-1}\big\}, \\ V_{ij}^+(m) &= \big\{x \in \operatorname{int}(P_m) \mid c_i x_i^{m-1} > c_j x_j^{m-1}\big\}, \\ V_{ij}^+(m) &= \big\{x \in \operatorname{int}(P_m) \mid c_i x_i^{m-1} < c_j x_j^{m-1}\big\} \end{aligned} \tag{31}$$

for any $i, j \in [1, n]$.

COROLLARY 4.1. *$V_{ij}^0(m)$, $V_{ij}^{\pm}(m)$ are invariant manifolds for* (1).

Proof. The result immediately follows from (30). ∎

Observe now that for $m < 3$ the invariant manifold P_m is compact. Since by Proposition 4.2, $q(x)$ is a Lyapunov function for (1) on $\operatorname{int}(P_m)$, we can conclude that for any $x \in \operatorname{int}(P_m)$ the solution $x(t)$ to (1) such that $x(0) = x$ converges to fixed points of (1) for $t \to \pm\infty$ (see, e.g., [11]). Those fixed points were described in Proposition 4.1 for $m \neq 1$. We introduce natural notations

$$x\big(J_{+\infty}(x)\big) = x(+\infty), \tag{32}$$

$$x\big(J_{-\infty}(x)\big) = x(-\infty); \tag{33}$$

see (25), (26). Given $x \in \operatorname{int}(P_m)$, denote by

$$\beta = \min\{c_i x_i^{m-1} \mid i \in [1, n]\}, \tag{34}$$

$$\gamma = \max\{c_i x_i^{m-1} \mid i \in [1, n]\}, \tag{35}$$

$$J_{\min}(x) = \{i \in [1, n] \mid c_i x_i^{m-1} = \beta\}, \tag{36}$$

$$J_{\max}(x) = \{i \in [1, n] \mid c_i x_i^{m-1} = \gamma\}. \tag{37}$$

LEMMA 4.1. *Let $m < 3$, $x(t)$ be a solution to* (1) *such that $x(0) = x \in \operatorname{int}(P_m)$. Then*

$$\lim_{t \to \pm\infty} q(x(t)) = \alpha\big(J_{\pm\infty}(x)\big). \tag{38}$$

Proof. Indeed,

$$q(x) = \sum_{i=1}^{n} x_i^{3-m} c_i x_i^{m-1}.$$

We know from Proposition 4.1 that

$$c_i x_i^{m-1}(\pm\infty) = \alpha\big(J_{\pm\infty}(x)\big), \qquad i \in J_{\pm\infty}(x),$$

$$\sum_{i=1}^{n} x_i^{3-m}(\pm\infty) = 1.$$

The result follows. ∎

LEMMA 4.2. *Given $x \in \operatorname{int}(P_m)$, $m > 3$, we have*

(a) *for any $i \in J_{\min}(x)$, $x_i(t)$ is a monotonically increasing function of t;*

(b) *for any $i \in J_{\max}(x)$, $x_i(t)$ is a monotonically decreasing function of t.*

Proof. Let $i \in J_{\min}(x)$. We have by (1)

$$\frac{dx_i(t)}{dt} = x_i(t)\Big(q(x(t)) - c_i x_i(t)^{m-1}\Big).$$

Further,

$$\begin{aligned} q(x(t)) &= \sum_{k=1}^{n} c_k x_k^{m-1}(t) x_k^{3-m}(t) \\ &\geq c_i x_i^{m-1}(t) \sum_{k=1}^{n} x_k^{3-m}(t) \\ &= c_i x_i^{m-1}(t). \end{aligned}$$

To see this we use Corollary 4.1. Indeed, since $i \in J_{\min}(x)$, $c_k x_k^{m-1}(0) = c_k x_k^{m-1} \geq c_i x_i^{m-1} = c_i x_i^{m-1}(0)$. Hence, $c_k x_k^{m-1}(t) \geq c_i x_i^{m-1}(t)$ for any t and any $k \in [1, n]$. Thus $dx_i(t)/dt \geq 0$ for every $t \in \mathbf{R}$. The proof of (b) is quite similar. ∎

COROLLARY 4.2. *Let $m < 3$. Then for any $x \in \operatorname{int}(P_m)$*

$$J_{\min}(x) \subset J_{+\infty}(x), \qquad J_{\max}(x) \subset J_{-\infty}(x).$$

Proof. Let $x(t)$ be the solution to (1) such that $x(0) = x$. Then for $i \in J_{\min}(x)$ we have by Lemma 4.2

$$x_i(+\infty) \geq x_i(0) > 0.$$

Hence $i \in J_{+\infty}(x)$. See (32) and Proposition 4.1. Similarly, for $i \in J_{\max}(x)$, $x_i(-\infty) > 0$ and hence $i \in J_{-\infty}(x)$. ∎

We are now in a position to describe $J_{\pm\infty}(x)$ for any $x \in \operatorname{int}(P_m)$. Let $\varnothing$ be the empty set.

THEOREM 4.1. *Let $1 < m < 3$, $x \in \operatorname{int}(P_m)$ and $x(t)$ be the solution to* (1) *such that $x(0) = x$. We have*

(a) *If $J_+ \neq \varnothing$, $J_- \neq \varnothing$, then*

$$J_{+\infty}(x) = J_{\min}(x), \qquad J_{-\infty}(x) = J_{\max}(x).$$

(b) *If $J_+ = [1, n]$, $J_- = \varnothing$, then*

$$J_{+\infty}(x) = [1, n], \qquad J_{-\infty}(x) = J_{\max}(x).$$

(c) *If $J_+ = \varnothing$, $J_- = [1, n]$, then*

$$J_{+\infty}(x) = J_{\min}(x), \qquad J_{-\infty}(x) = [1, n].$$

Proof. Consider the case (a). By Corollary 4.2 we know that $J_{\min}(x) \subset J_{+\infty}(x)$. In particular, since $J_{\min}(x) \subset J_-$, we conclude that $\alpha(J_{+\infty}(x)) < 0$; see (27). Suppose that $j \in J_{+\infty}(x) \setminus J_{\min}(x)$, $i \in J_{\min}(x)$. Then

$$c_j x_j^{m-1}(+\infty) = c_i x_i^{m-1}(+\infty) = \alpha\big(J_{+\infty}(x)\big). \tag{39}$$

Hence by (38)

$$\lim_{t \to +\infty} \Big(c_i x_i^{m-1}(t) + c_j x_j^{m-1}(t) - q(x(t))\Big) = \alpha\big(J_{+\infty}(x)\big) < 0. \tag{40}$$

We know that

$$c_j x_j^{m-1}(t) > c_i x_i^{m-1}(t) \tag{41}$$

for all t. We can conclude from (30), (40), (41) that

$$\frac{d}{dt}\Big(c_j x_j^{m-1}(t) - c_i x_i^{m-1}(t)\Big) > 0 \tag{42}$$

for sufficiently large positive t. But then (39), (41), (42) lead to a contradiction. Hence, $J_{+\infty}(x) = J_{\min}(x)$. The proof of $J_{-\infty}(x) = J_{\max}(x)$ is quite similar.

Consider the case (b). By Corollary 4.2, $J_{\min}(x) \subset J_{+\infty}(x)$. For any $j \in [1,n]$, $i \in J_{\min}(x)$

$$c_j x_j^{m-1}(t) \geq c_i x_i^{m-1}(t)$$

by Corollary 4.1. Hence

$$c_j x_j^{m-1}(-\infty) \geq c_i x_i^{m-1}(+\infty) = \alpha\big(J_{+\infty}(x)\big) > 0.$$

Hence $x_j(+\infty) > 0$ and $j \in J_{+\infty}(x)$. The proof of $J_{-\infty}(x) = J_{\max}(x)$ is exactly the same as the proof in (a). The proof of (c) is quite similar to the proof of (b). ∎

THEOREM 4.2. *Let $m < 1$, $x \in \operatorname{int}(P_m)$, and $x(t)$ be the solution to* (1) *such that $x(0) = x$. We have*

(a) *If $J_+ \neq \varnothing$, $J_- \neq \varnothing$, then*

$$J_{+\infty}(x) = J_-, \qquad J_{-\infty}(x) = J_+.$$

(b) *If $J_+ = [1,n]$, $J_- = \varnothing$, then*

$$J_{+\infty}(x) = J_{\min}(x), \qquad J_{-\infty}(x) = [1,n].$$

(c) *If $J_+ = \varnothing$, $J_- = [1,n]$, then*

$$J_{+\infty}(x) = [1,n], \qquad J_{-\infty}(x) = J_{\max}(x).$$

Proof. Consider the case (a). By Corollary 4.2, $J_{\min}(x) \subset J_{+\infty}(x)$. Since $J_- \neq \emptyset$, $J_{\min}(x) \subset J_-$. Hence, $\alpha(J_{+\infty}(x)) < 0$. If $j \in J_-$, $i \in J_{\min}(x)$, then by Corollary 4.1, $c_j x_j^{m-1}(t) \geq c_i x_i^{m-1}(t)$ for any t. Note that $c_j < 0$ and $m < 1$. If $x_j(t) \to 0$, $t \to +\infty$, then $c_j x_j^{m-1}(t) \to -\infty$. This leads to a contradiction. Thus $x_j(+\infty) > 0$ and $J_{+\infty}(x) \supset J_-$. On the other hand if $j \in J_+$, then $c_j x_j^{m-1}(t) \geq 0$ for all t. Hence $c_j x_j^{m-1}(+\infty) > \alpha(J_{+\infty}(x))$. Thus $x_j(+\infty) = 0$, i.e., $J_{+\infty}(x) = J_-$. The proof of $J_{-\infty}(x) = J_+$ is quite similar.

Consider the case (b). By Corollary 4.2, $J_{\min}(x) \subset J_{+\infty}(x)$. Hence $\alpha(J_{+\infty}(x)) > 0$. Suppose that $j \in J_{+\infty}(x) \setminus J_{\min}(x)$, $i \in J_{\min}(x)$. Then

$$c_j x_j^{m-1}(+\infty) = c_i x_i^{m-1}(+\infty) = \alpha\big(J_{+\infty}(x)\big)$$

$$\lim_{t \to +\infty} \Big(c_j x_j^{m-1}(t) + c_i x_i^{m-1}(t) - q(x(t))\Big) = \alpha\big(J_{+\infty}(x)\big) > 0 \quad (43)$$

by Lemma 4.1 and

$$c_j x_j^{m-1}(t) > c_i x_i^{m-1}(t) \quad (44)$$

for all t by Corollary 4.1. Hence, by (30)

$$\frac{d}{dt}\Big(c_j x_j^{m-1}(t) - c_i x_i^{m-1}(t)\Big) > 0$$

for sufficiently large t since $m < 1$. This contradicts (43), (44). Hence, $J_{+\infty}(x) = J_{\min}(x)$. Further, by Corollary 4.2, $J_{-\infty}(x) \supset J_{\max}(x)$. In particular, $\alpha(J_{-\infty}(x)) > 0$. Let $i \in J_{\max}(x)$, $j \in [1, n]$. We have

$$c_i x_i^{m-1}(t) \geq c_j x_j^{m-1}(t) \quad (45)$$

for all t. If $x_j(t) \to 0$, $t \to -\infty$, then $c_j x_j^{m-1}(t) \to +\infty$, $t \to -\infty$. This contradicts (45). Hence $x_j(-\infty) > 0$ and $j \in J_{-\infty}(x)$. This completes the proof of (b). The proof of (c) is quite similar to the proof of (b). ∎

Remark 4.1. The case $m = 2$ (Karmarkar's flow) was considered in [5]. The case $m = 1$ (Moser's flow) is easily treated through explicit formulas for the solution (23).

Remark 4.2. The case where some $c_i = 0$ is more complicated. Suppose that $c_i \neq 0$, $i \leq n_i$, $c_i = 0$, $n_i < i \leq n$. Then the first n_1 equations decouple from the entire system and have the form of the original system of equations.

Remark 4.3. In the case where some components of the initial vector $x(0)$ are equal to zero, they remain equal to zero and the problem is reduced to the case $x(0) \in \text{int}(P_m)$ with a smaller number of variables.

Remark 4.4. In the case $m > 3$ the phase space is not compact. The phase portrait is rather trivial.

5. A DYNAMICAL SYSTEM HAVING CUBIC NONLINEARITY

As was shown in Section 2 the dynamical systems of Moser−Karmarkar type (1) with $m = 3$ cannot admit the invariant manifold of the form (2) except for the case $n = 2$ and $c_1 + c_2 = 0$. Here we show that a slight modification of (1) with $m = 3$ has a property similar to that of the Karmarkar case where $m = 2$. Let us consider the dynamical system

$$\frac{dx_j}{dt} = -c_j x_j^3 + \frac{1}{n} x_j \sum_{k=1}^{n} c_k x_k^2, \qquad j = 1, 2, \dots, n \tag{46}$$

which has cubic nonlinearity. Define

$$I(t) = \log \prod_{j=1}^{n} x_j(t). \tag{47}$$

By a direct calculation we see that $I(t)$ is a first integral of (46), namely, $I(t) = c$ for some constant c. If we set $y_j = x_j^2$, we can rewrite (46) in a form similar to that of the Lotka−Volterra system.

$$\frac{dy_j}{dt} = -2y_j \left(c_j y_j = \frac{1}{n} \sum_{k=1}^{n} c_k y_k \right).$$

It is known [14] that the quantity $\prod_{j=1}^{n} y_j$ is one of the first integrals of a cyclic Lotka−Volterra equation of odd species. Set $z_j = \log x_j$. Then we have

$$\frac{dz_j}{dt} = -c_j e^{2z_j} + \frac{1}{n} z_j \sum_{k=1}^{n} c_k e^{2z_k}. \tag{48}$$

Let us suppose $z_j(0) > 0$. Then (48) is a gradient system on the simplex $\sum_{j=1}^{n} z_j = n$ for the potential $U = \frac{1}{2} \sum_{k=1}^{n} c_k e^{2z_k} (\sum_{k=1}^{n} z_k)^{-1}$ relative to the Euclidean metric $(g_{ij}) = n^{-1}(\delta_{ij})$. A Lax representation of (48) is

$$\frac{dL(z)}{dt} = [[L(z), D], L(z)], \tag{49}$$

where

$$L(z) = \sqrt{z}\,\sqrt{z}^{\top}, \qquad \sqrt{z} = (\sqrt{z}_1, \dots, \sqrt{z}_n)^{\top},$$

$$D = \frac{1}{2n} \operatorname{diag}(u_1(z), \dots, u_n(z)), \qquad u_i(z) = \frac{c_i}{z_i} e^{2z_i} - \frac{1}{n} \sum_{k=1}^{n} c_k e^{2z_k}. \tag{50}$$

Next we show how to integrate (46) with the help of (47). Let us introduce

$$T(t) = \exp\left(\frac{2}{n}\int_0^t \sum_{k=1}^{n} c_k x_k^2(s)\, ds\right).$$

From the dynamical system (46) we derive $d(x_j^{-2}T)/dt = 2c_jT$. As in Section 3 we get

$$x_j(t) = \left(\frac{dP/dt}{x_j^{-2}(0) + 2c_jP}\right)^{1/2}, \tag{51}$$

where $P(t) = \int_0^t T(s)\, ds$. Substituting $x_j(t)$ into the first integral $\log \prod_{j=1}^{n} x_j = c$ we derive the differential equation

$$\frac{dP}{dt} = \left(e^{2c}\prod_{j=1}^{n}\left(x_j^{-2}(0) + 2c_jP\right)\right)^{1/n}, \tag{52}$$

which is somewhat different from (21). It is shown here that (46) is also an explicitly solvable gradient system having the Lax representation (49).

6. CONCLUDING REMARKS

We have considered a family of explicitly solvable gradient flows parametrized by the real number m. One can distinguish the following cases: (i) $m < 1$, (ii) $1 < m < 3$, (iii) $m > 3$. The topological structure of the phase portrait changes at $m = 1$ (Moser's flow) and $m = 3$. The region $1 < m < 3$ corresponds to Karmarkar-type flows ($m = 2$ is the Karmarkar flow itself). There are explicit formulas for solutions in all cases. The phase portrait is completely described for $m < 3$. The Lax representation in the double-bracket form is established for all m.

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